Determinant (Answers)

1. **(a)** $D = 2\times\begin{vmatrix}1&1&1\\3&1&1\\4&2&1\end{vmatrix} = 2\times\begin{vmatrix}0&0&1\\2&0&1\\3&1&1\end{vmatrix} = 2\times\begin{vmatrix}2&0\\3&1\end{vmatrix} = 2\times 2 = \underline{\underline{4}}$

(b) C_1 and C_2 are in ratio, therefore D = 0.

(c) Use Sarrus rule to expand the given determinant, D = $\underline{\underline{2abc}}$.

(d) $D \overset{R_3 \to R_1 + R_2 + R_3}{=} \begin{vmatrix}b+c & a-b & a\\ c+a & b-c & b\\ 2(a+b+c) & 0 & a+b+c\end{vmatrix} = (a+b+c)\begin{vmatrix}b+c & a-b & a\\ c+a & b-c & b\\ 2 & 0 & 1\end{vmatrix}$

$= \underline{\underline{-(a+b+c)(a^2+b^2+c^2-bc-ca-ab)}}$

(e) $D \overset{C_3 \to C_3 - C_1}{=} \begin{vmatrix}4&3&1\\1&6&1\\7&-2&1\end{vmatrix} \overset{R_3 \to R_3 - R_1}{=} \begin{vmatrix}4&3&1\\1&6&0\\3&-5&0\end{vmatrix} = \begin{vmatrix}1&6\\3&-5\end{vmatrix} = \underline{\underline{-23}}$

(f) By Sarrus rule of expansion, result follows.

2. **(a)** L.H.S. = $\begin{vmatrix}a&b&c\\p&q&r\\x&y&z\end{vmatrix}\times\begin{vmatrix}0&1&1\\1&0&1\\1&1&0\end{vmatrix}$ = R.H.S.

(b) $D \underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - R_2}{=}} \begin{vmatrix}0 & a-b & a^2-b^2\\ 0 & b-c & b^2-c^2\\ 1 & c & c^2\end{vmatrix} = \begin{vmatrix}a-b & a^2-b^2\\ b-c & b^2-c^2\end{vmatrix} = (a-b)(b^2-c^2)-(b-c)(a^2-b^2)$

$= (a-b)(b-c)\{(b+c)-(a+b)\} = \underline{\underline{(a-b)(b-c)(c-a)}}$

(c) $D = \begin{vmatrix}1&a&a^3\\1&b&b^3\\1&c&c^3\end{vmatrix} \overset{C_3 \to C_3 + (ab+bc+ca)C_2 - abcC_1}{=} \begin{vmatrix}1&a&a^2(a+b+c)\\1&b&b^2(a+b+c)\\1&c&c^2(a+b+c)\end{vmatrix}$

= (a + b + c) × determinant in 2(b)

= $\underline{\underline{(a+b+c)(a-b)(b-c)(c-a)}}$

(d) $D \overset{R_3 \to R_3 + (x+y+z)R_1 - R_2}{=} \begin{vmatrix}x & y & z\\ x^2 & y^2 & z^2\\ yz+zx+xy & yz+zx+xy & yz+zx+xy\end{vmatrix}$

= $\underline{\underline{(yz+zx+xy)(x-y)(y-z)(z-x)}}$

by **(i)** taking out (yz + z x + xy) in R_3 ;

(ii) interchange R_2 with R_3 and then R_1 with R_2 ;

(iii) result in 2 (c) where a = x , b = y , c = z .

(e) D = $-8abc + 2(b+c)(c+a)(a+b) + 2a(b+c)^2 + 2b(c+a)^2 + 2c(a+b)^2$

= $2(b+c)(c+a)(a+b) + 2a(b+c)^2 - 8abc + 2b(c^2+2ca+a^2) + 2c(a^2+2ab+b^2)$

= $2(b+c)(c+a)(a+b) + 2a(b+c)^2 + 2bc^2 + 2ba^2 + 2ca^2 + 2cb^2$

= $2(b+c)(c+a)(a+b) + 2a(b+c)^2 + 2a^2(b+c) + 2bc(b+c)$

= $2(b+c)(c+a)(a+b) + 2(b+c)\{a(b+c) + a^2 + 2bc\}$

= $2(b+c)(c+a)(a+b) + 2(b+c)(c+a)(a+b)$

= $\underline{\underline{4(b+c)(c+a)(a+b)}}$

3. **(a)** $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \overset{R_1 \to R_1 + R_2 + R_3}{=} \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} = 0$,since $1+\omega+\omega^2 = \dfrac{1-\omega^3}{1-\omega} = \dfrac{0}{\omega} = 0$.

(b) D $= 1 + \omega^6 + \omega^6 - \omega^4 - \omega^2 - \omega^6 = 1 + 1 + 1 - \omega - \omega^2 - 1$ $(\omega^3 = 1)$

$= 3 - (\omega^2 + \omega + 1) = 3 - 0 = \underline{\underline{3}}$

4. $D \overset{R_3 \to R_3 + 2R_2}{=} \begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = -(a+b+c)\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$

$= \underline{\underline{-(a + b + c)^2 (a - b) (b - c) (c - a)}}$

5. $D \overset{R_1 \to R_1 + R_2 + R_3}{=} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$\overset{\substack{C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1}}{=} \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} = (a+b+c)\begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix}$

$= \underline{\underline{(a + b + c)^3}}$

6. $D \overset{C_2 \to C_2 + C_1}{=} \begin{vmatrix} a & a+b+c & a^2 \\ b & a+b+c & b^2 \\ c & a+b+c & c^2 \end{vmatrix} = (a+b+c)\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = -(a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= \underline{\underline{-(a + b + c) (a - b) (b - c) (c - a)}}$, by 2 (b)

7. $D \overset{C_2 \to C_2 - 2C_1 - 2C_3}{=} \begin{vmatrix} a^2 & -(a^2+b^2+c^2) & bc \\ b^2 & -(a^2+b^2+c^2) & ca \\ c^2 & -(a^2+b^2+c^2) & ab \end{vmatrix} = -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$

$= -abc(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & 1/a \\ b^2 & 1 & 1/b \\ c^2 & 1 & 1/c \end{vmatrix} = -(a^2+b^2+c^2)\begin{vmatrix} a^3 & a & 1 \\ b^3 & b & 1 \\ c^3 & c & 1 \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= \underline{\underline{(a^2 + b^2 + c^2) (a + b + c) (a - b)(b - c)(c - a)}}$, by 2(c)

8. $\text{L.H.S.} = \dfrac{1}{abc}\begin{vmatrix} abc & abc & abc \\ bc(c-b) & ca(a-c) & ab(b-a) \\ b^2c & c^2a & a^2b \end{vmatrix} = \dfrac{(bc)(ca)(ab)}{abc}\begin{vmatrix} a & b & c \\ c-b & a-c & b-a \\ b & c & a \end{vmatrix}$

$\overset{R_2 \to R_2 + R_3}{=} abc\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \underline{\underline{abc\,(a^3 + b^3 + c^3 - 3abc)}}$

9. $D(x) = \begin{vmatrix} 1 & -x & -x & -x \\ 0 & r_1+x & a+x & a+x \\ 0 & b+x & r_2+x & a+x \\ 0 & b+x & b+x & r_3+x \end{vmatrix} \overset{\substack{R_i \to R_i + R_1 \\ (i=1,2,3)}}{=} \begin{vmatrix} 1 & -x & -x & -x \\ 1 & r_1 & a & a \\ 1 & b & r_2 & a \\ 1 & b & b & r_3 \end{vmatrix}$

$= D(0) + x\begin{vmatrix} 1 & a & a \\ 1 & r_2 & a \\ 1 & b & r_3 \end{vmatrix} + x\begin{vmatrix} r_1 & 1 & a \\ b & 1 & a \\ b & 1 & r_3 \end{vmatrix} + x\begin{vmatrix} r_1 & a & 1 \\ b & r_2 & 1 \\ b & b & 1 \end{vmatrix}$, by expansion in R_1.

$$= D(0) + x\begin{vmatrix} 1 & 1 & 1 & 0 \\ r_1 & a & a & 1 \\ b & r_2 & a & 1 \\ b & b & r_3 & 1 \end{vmatrix} = D(0) + x\,\Delta \quad \ldots. (*)$$

By substitution, it can be seen that $D(-b) = (r_1 - b)(r_2 - b)(r_3 - b) = f(b)$

and $D(-a) = (r_1 - a)(r_2 - a)(r_3 - a) = f(a)$

By (*), $f(b) = D(-b) = D(0) - b\,\Delta$ …. (1)

$f(a) = D(-a) = D(0) - a\,\Delta$ …. (2)

Solving (1) and (2) for D(0), result follows.

10. $D(a,b,c) \overset{R_1 \to R_1 + R_2}{=} \begin{vmatrix} ab+bc+ca & ab+bc+ca & ab+bc+ca \\ bc & ca & ab \\ 1 & 1 & 1 \end{vmatrix} = 0$, since R_1 and R_3 are in ratio.

Therefore a, b, c, (b – c), (c – a), (a – b) are factors of 0 .

11. Writing $a = \sin x$, $b = \sin y$, $c = \sin z$; $p = \cos x$, $q = \cos y$, $r = \cos z$

$$D = \begin{vmatrix} a & b & c \\ 2ap & 2bq & 2cr \\ a(3-4a^2) & b(3-4b^2) & c(3-4c^2) \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ 2p & 2q & 2r \\ 3-4a^2 & 3-4b^2 & 3-4c^2 \end{vmatrix}$$

$$\overset{\substack{R_2 \to R_2 - R_3 \\ R_3 \to R_3 \to R_1}}{=} abc\begin{vmatrix} 1 & 0 & 0 \\ 2p & 2(q-r) & 2(r-p) \\ 3-4a^2 & 4(c^2-b^2) & 4(a^2-c^2) \end{vmatrix} = 8abc\begin{vmatrix} q-r & r-p \\ (q-r)(q+r) & (r-p)(r+p) \end{vmatrix}$$

$= 8abc(q - r)(r - p)(p - q)$

$= \underline{8 \sin x \sin y \sin z\, (\cos y - \cos z)(\cos z - \cos x)(\cos x - \cos y)}$

12. **Method 1**

$$\frac{dD}{d\theta} = \begin{vmatrix} -\sin(\theta+x) & \sin(\theta+x) & 1 \\ -\sin(\theta+y) & \sin(\theta+y) & 1 \\ -\sin(\theta+z) & \sin(\theta+z) & 1 \end{vmatrix} + \begin{vmatrix} \cos(\theta+x) & \cos(\theta+x) & 1 \\ \cos(\theta+y) & \cos(\theta+y) & 1 \\ \cos(\theta+z) & \cos(\theta+z) & 1 \end{vmatrix} + \begin{vmatrix} \cos(\theta+x) & \sin(\theta+x) & 0 \\ \cos(\theta+y) & \sin(\theta+y) & 0 \\ \cos(\theta+z) & \sin(\theta+z) & 0 \end{vmatrix} = 0$$

$\therefore$ D is independent of θ .

Method 2

$$\begin{vmatrix} \cos(\theta+x) & \sin(\theta+x) & 1 \\ \cos(\theta+y) & \sin(\theta+y) & 1 \\ \cos(\theta+z) & \sin(\theta+z) & 1 \end{vmatrix} \overset{\substack{R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3}}{=} \begin{vmatrix} -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) & 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) & 0 \\ -2\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{y-z}{2}\right) & 2\cos\left(\frac{y-z}{2}\right)\sin\left(\frac{y-z}{2}\right) & 0 \\ \cos(\theta+z) & \sin(\theta+z) & 1 \end{vmatrix}$$

$$= -4\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{y-z}{2}\right)\begin{vmatrix} \sin\left(\frac{x+y}{2}\right) & \cos\left(\frac{x+y}{2}\right) \\ \sin\left(\frac{y+z}{2}\right) & \cos\left(\frac{y+z}{2}\right) \end{vmatrix} = -4\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{y-z}{2}\right)\sin\left(\frac{z-x}{2}\right)$$

which is independent of θ.

13. Let f(z) = given determinant.

$f(\,(1 + a^2)\sin(\pi/6)\,) = 0$, by substituting into the given determinant, R_1 and R_3 are equal.

$f(\,(1 + a^2)\sin(\pi/3)\,) = 0$, by substituting into the given determinant, C_1 and C_3 are in ratio.

$f(\,(1 + a^2)\,x\,)$ is a determinant of degree 3.

It can be easily seen from the determinant that : Coefficient of x^2 – term = 0.

Let α be the third root, Sum of roots = $\alpha + \sin(\pi/6) + \sin(\pi/3) = 0$

$$\therefore\ \ \alpha = -\frac{1}{2} - \frac{\sqrt{3}}{2}\ .$$

14. $$\Delta = \frac{1}{-2\cos x}\begin{vmatrix} 1 & -2a\cos x & a^2 \\ \cos(n-1)x & -2\cos nx\cos x & \cos(n+1)x \\ \sin(n-1)x & -2\sin nx\cos x & \sin(n+1)x \end{vmatrix}$$

$$= \frac{1}{-2\cos x}\begin{vmatrix} 1 & -2a\cos x & a^2 \\ \cos(n-1)x & -[\cos(n+1)x + \cos(n-1)x] & \cos(n+1)x \\ \sin(n-1)x & -[\sin(n+1) + \sin(n-1)x] & \sin(n+1)x \end{vmatrix}$$

$$\overset{C_2 \to C_2 + C_1 + C_3}{=} \frac{1}{-2\cos x}\begin{vmatrix} 1 & 1 - 2a\cos x + a^2 & a^2 \\ \cos(n-1)x & 0 & \cos(n+1)x \\ \sin(n-1)x & 0 & \sin(n-1)x \end{vmatrix}$$

$$= \frac{1 - 2a\cos x + a^2}{2\cos x}\begin{vmatrix} \cos(n-1)x & \cos(n+1)x \\ \sin(n-1)x & \sin(n-1)x \end{vmatrix}$$

$$= \frac{1 - 2a\cos x + a^2}{2\cos x}\sin 2x = \left(1 - 2a\cos x + a^2\right)\sin x$$

15. $$\text{L.H.S.} = \begin{vmatrix} 1 & \cos A & \cos A \\ 1 & \cos B & \cos B \\ 1 & \cos C & \cos C \end{vmatrix} + \begin{vmatrix} 1 & -\sin A & \sin A \\ 1 & -\sin B & \sin B \\ 1 & -\sin C & \sin C \end{vmatrix} + \begin{vmatrix} 1 & -\sin A & \cos A \\ 1 & -\sin B & \cos B \\ 1 & -\sin C & \cos C \end{vmatrix} + \begin{vmatrix} 1 & \cos A & \sin A \\ 1 & \cos B & \sin B \\ 1 & \cos C & \sin C \end{vmatrix}$$

= 0 + 0 + 0 + R.H.S. = R.H.S.

16. $$\Delta = \begin{vmatrix} \sin B & \sin C \\ \cos B & \cos C \end{vmatrix} - \begin{vmatrix} \sin A & \sin C \\ \cos A & \cos C \end{vmatrix} + \begin{vmatrix} \sin A & \sin B \\ \cos A & \cos B \end{vmatrix} = \sin(A - B) + \sin(B - C) + \sin(C - A)$$

$$\Delta \underset{C_3 \to C_3 - C_1}{\overset{C_2 \to C_2 - C_3}{=}} \begin{vmatrix} 1 & 0 & 0 \\ \sin A & \sin B - \sin C & \sin C - \sin A \\ \cos A & \cos B - \cos C & \cos C - \cos A \end{vmatrix} = \begin{vmatrix} 2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right) & 2\cos\left(\frac{C+A}{2}\right)\sin\left(\frac{C-A}{2}\right) \\ -2\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right) & -2\sin\left(\frac{C+A}{2}\right)\sin\left(\frac{C-A}{2}\right) \end{vmatrix}$$

$$= -4\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{C-A}{2}\right)\begin{vmatrix} \cos\left(\frac{B+C}{2}\right) & \cos\left(\frac{C+A}{2}\right) \\ \sin\left(\frac{B+C}{2}\right) & \sin\left(\frac{C+A}{2}\right) \end{vmatrix} = -4\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{C-A}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

17. Expansion on both sides gives result. $|A|\,|B| = |AB|$ for 2×2 determinants.

18. D = first given determinant = D′ (the transpose) = (b – c)(c – a)(a – b) , by 2(b)

Therefore, second given determinant = DD′ = $[(b - c)(c - a)(a - b)]^2$

19. Interchanging C_2 and C_3 gives the first result.

Let the first given determinant be D .

$$\text{then}\quad D^2 = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}\begin{vmatrix} -a & -c & -b \\ c & b & a \\ b & a & c \end{vmatrix} = \begin{vmatrix} 2bc-a^2 & b^2 & c^2 \\ b^2 & 2ab-c^2 & a^2 \\ c^2 & a^2 & 2ca-b^2 \end{vmatrix}$$

= last determinant, by interchanging C_2 and C_3 , then R_2 and R_3 .

20. The other factor is $\begin{vmatrix} b & c & a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = 0$.

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}\begin{vmatrix} b & c & a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2ab & ac+b^2 & bc+a^2 \\ bc+a^2 & ab+c^2 & 2ac \\ ab+b^2 & 2bc & ab+c^2 \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{=} \begin{vmatrix} 2ab & ac+b^2 & bc+a^2 \\ ab+b^2 & 2bc & ab+c^2 \\ bc+a^2 & ab+c^2 & 2ac \end{vmatrix} = 0$$

21. $$\begin{vmatrix} s_0 & s_2 & s_3 \\ s_4 & s_6 & s_7 \\ s_6 & s_8 & s_9 \end{vmatrix} = \left|\begin{pmatrix} 1 & 1 & 1 \\ \alpha^4 & \beta^4 & \gamma^4 \\ \alpha^6 & \beta^6 & \gamma^6 \end{pmatrix}\begin{pmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{pmatrix}\right| = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^4 & \beta^4 & \gamma^4 \\ \alpha^6 & \beta^6 & \gamma^6 \end{vmatrix}\begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

22. $D_3 = D_3' = D_1 \times D_2 = D_1 \times 0 = 0$.

23. $$|A| \underset{C_2 \to C_2 - C_1}{\overset{C_3 \to C_3 - C_2}{=}} \begin{vmatrix} 1 & 0 & 0 \\ a+b & c-a & a-b \\ ab & b(c-a) & c(a-b) \end{vmatrix} = \begin{vmatrix} c-a & a-b \\ b(c-a) & c(a-b) \end{vmatrix} = (c-a)(a-b)\begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}$$

$= -(c-a)(a-b)(b-c)$

$$X = \begin{pmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{pmatrix}$$

det B = (det x) (det A) = $[-(x-y)(y-z)(z-x)][-(a-b)(b-c)(c-a)] = (x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

24. **(a)** $$\Delta \times \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} aA+hH+gG & aH+hB+gF & aG+hF+gC \\ hA+bH+fG & hH+bF+fF & hG+bF+fC \\ gA+fH+cG & gH+fB+cF & gG+fF+cC \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

Therefore, if $\Delta \neq 0$, cancel Δ on both sides, result follows.

if $\Delta = 0$, both sides = 0 and result follows.

(b) $$\Delta \times \begin{vmatrix} 1 & H & G \\ 0 & B & F \\ 0 & F & C \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ h & \Delta & 0 \\ g & 0 & \Delta \end{vmatrix} = a\Delta^2$$

$\therefore \quad \Delta (BC - F^2) = a\Delta^2$.

Result follows for both $\Delta \neq 0$.

25. Consider $\Delta \times \begin{vmatrix} A & 0 & G \\ H & 0 & F \\ G & 1 & C \end{vmatrix}$. Similar to 24 (b).

26. Multiply R_1 by a , R_2 by b , R_3 by c in the given equation, we get

$$\begin{vmatrix} a & a^2+abcx & bc \\ b & b^2+abcx & ca \\ c & c^2+abcx & ab \end{vmatrix}=0 \quad\Rightarrow\quad \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}+abcx\begin{vmatrix} a & 1 & bc \\ b & 1 & ca \\ c & 1 & ab \end{vmatrix}=0$$

In the second determinant in the last equation, use $C_3 \to C_3 - (a + b + c)\, C_1 + (ab + bc + ca)C_2$,

then evaluate by 2(b).

The first determinant can be evaluated by 2(c).

We get $(a - b)(b - c)(c - a)\,(ab + bc + ca) - abcx\,(a - b)(b - c)(c - a) = 0$.

Therefore $\underline{\underline{x=\dfrac{ab+bc+ca}{abc}}}$

27. $$\begin{vmatrix} 1 & \sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ 0 & 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ 0 & \sin^2\theta & 1+\cos^2\theta & 4\sin 2\theta \\ 0 & \sin^2\theta & \cos^2\theta & 1+4\sin 2\theta \end{vmatrix}=0$$

Use $R_i \to R_i - R_1$, $i = 2, 3, 4$.

$$\begin{vmatrix} 1 & \sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}=0 \quad \underset{=}{C_1 \to \sum_{i=1}^{4} C_i} \quad \begin{vmatrix} 1+\sin^2\theta+\cos^2\theta+4\sin 2\theta & \sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}=0$$

Therefore $1 + \sin^2\theta + \cos^2\theta + 4\sin 2\theta = 0$

$2 + 4\sin 2\theta = 0$, $2\theta = n\pi + (-1)^n(-\pi/6)$

$\underline{\underline{\theta=\dfrac{1}{2}n\pi+(-1)^n\left(-\dfrac{\pi}{12}\right)}}$, where n is an integer.

28. **<u>Method 1</u>**

$$\begin{vmatrix} x & x^2 & a^3 \\ b & b^2 & a^3 \\ c & c^2 & a^3 \end{vmatrix}-\begin{vmatrix} x & x^2 & x^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=0 \quad\Rightarrow\quad a^3\begin{vmatrix} x & x^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}-xbc\begin{vmatrix} 1 & x & x^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$$

$a^3 (x - b)(b - c)(c - x) - xbc\,(x - b)(b - c)(c - x) = 0$

$(b - c)\,(x - b)\,(c - x)(a^3 - xbc) = 0$

$\underline{\underline{x=b,c,\dfrac{a^3}{bc}}}$

<u>Method 2</u>

It can be seen that $x = b, x = c$ are roots of the given equation by Factor Theorem.

Degree of the equation = 3 and let α be the third root.

From the determinant, Coefficient of x^3 – term $= b^2c$

Coefficient of constant term $= - a^3b^2c$

Product of roots $= \alpha\, bc$ $= - (- a^3b^2c)/(b^2c)$, solve for α we have:

$\underline{\underline{x=b,c,\dfrac{a^3}{bc}}}$

29. $\Delta = a^3 + 2b^3 - 3ab^2 = (a + 2b)(a - b)^2 = 7(x + 1)(x - 2)$

Therefore , $x = -1$ or 2.

30. $$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} = \begin{vmatrix} s & p & p \\ p & s & p \\ p & p & s \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} x & z & y \\ y & x & z \\ z & y & x \end{vmatrix}$$

31. $\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, $\overrightarrow{OC} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$

Area of ABC = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

Area of parallelogram formed by OB and OC as adjacent sides = $\left|\overrightarrow{OB} \times \overrightarrow{OC}\right|$

Volume of parallelepiped $= \overrightarrow{OA} \bullet \left(\overrightarrow{OB} \times \overrightarrow{OC}\right) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

32. Area of triangle formed by (x , y) , (x_1 , y_1) , $(x_2 , y_2) = 0$

By no. 31 , result followed.

33. $L_i : A_i + B_i\, y + C_i = 0$, $i = 1, 2, 3$ are concurrent

$\Rightarrow$ L_3 passes through the intersection point of L_1 and L_2

$\Rightarrow$ $(x, y) = (\Delta_x / \Delta , \Delta_y / \Delta)$ satisfies L_3

$\Rightarrow$ Coefficient determinant = 0

The condition is not sufficient,

Consider 3 parallel lines which are not concurrent, the coefficient determinant = 0